The CNO Bi-Cycle
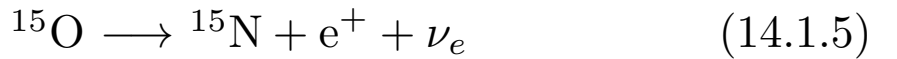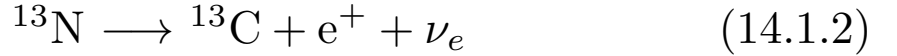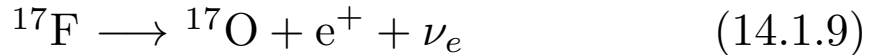
A second way of processing ^{1}H into ^{4}He is through a series of nuclear reactions involving the different species of Carbon, Nitrogen, and Oxygen. The principle reactions are as follows:

$$^{12}\text{C} + {}^1\text{H} \longrightarrow {}^{13}\text{N} + \gamma \tag{14.1.1}$$

$$^{13}\text{N} \longrightarrow {}^{13}\text{C} + e^+ + \nu_e \tag{14.1.2}$$

$$^{13}\text{C} + {}^1\text{H} \longrightarrow {}^{14}\text{N} + \gamma \tag{14.1.3}$$

$$^{14}\text{N} + {}^1\text{H} \longrightarrow {}^{15}\text{O} + \gamma \tag{14.1.4}$$

$$^{15}\text{O} \longrightarrow {}^{15}\text{N} + e^+ + \nu_e \tag{14.1.5}$$

$$^{15}\text{N} + {}^1\text{H} \longrightarrow {}^{12}\text{C} + {}^4\text{He} \tag{14.1.6}$$

or

$$^{15}\text{N} + {}^1\text{H} \longrightarrow {}^{16}\text{O} + \gamma \tag{14.1.7}$$

$$^{16}\text{O} + {}^1\text{H} \longrightarrow {}^{17}\text{F} + \gamma \tag{14.1.8}$$

$$^{17}\text{F} \longrightarrow {}^{17}\text{O} + e^+ + \nu_e \tag{14.1.9}$$

$$^{17}\text{O} + {}^1\text{H} \longrightarrow {}^{14}\text{N} + {}^4\text{He} \tag{14.1.10}$$

Note that the net sum of these reactions is

$$^{12}\text{C} + 4\,{}^1\text{H} \longrightarrow {}^{12}\text{C} + {}^4\text{He} + 2\beta^+ + 2\nu_e$$
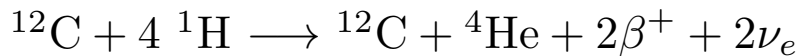
In other words, four protons are converted into two positrons, two electron neutrinos, and one ^{4}He nucleus; ^{12}C serves only as a catalyst. Moreover, since the cycle can begin with any of the intermediate produces, it will occur if any of them exists.

The differential equations for the full CNO bi-cyle are below. To simplify the notation, the number density symbol, N has been omitted.

$$\frac{d^{12}\mathrm{C}}{dt} = -\frac{^{12}\mathrm{C}}{\tau} + (1-\gamma)\,\frac{^{15}\mathrm{N}}{\tau} \tag{14.1.11}$$

$$\frac{d^{13}\mathrm{N}}{dt} = +\frac{^{12}\mathrm{C}}{\tau} - \frac{^{13}\mathrm{N}}{\tau_\beta} \tag{14.1.12}$$

$$\frac{d^{13}\mathrm{C}}{dt} = +\frac{^{13}\mathrm{N}}{\tau_\beta} - \frac{^{13}\mathrm{C}}{\tau} \tag{14.1.13}$$

$$\frac{d^{14}\mathrm{N}}{dt} = +\frac{^{13}\mathrm{C}}{\tau} - \frac{^{14}\mathrm{N}}{\tau} + \frac{^{17}\mathrm{O}}{\tau} \tag{14.1.14}$$

$$\frac{d^{15}\mathrm{O}}{dt} = +\frac{^{14}\mathrm{N}}{\tau} - \frac{^{15}\mathrm{O}}{\tau_\beta} \tag{14.1.15}$$

$$\frac{d^{15}\mathrm{N}}{dt} = +\frac{^{15}\mathrm{O}}{\tau_\beta} - (1-\gamma)\,\frac{^{15}\mathrm{N}}{\tau} - \gamma\,\frac{^{15}\mathrm{N}}{\tau} \tag{14.1.16}$$

$$\frac{d^{16}\mathrm{O}}{dt} = \gamma\,\frac{^{15}\mathrm{N}}{\tau} - \frac{^{16}\mathrm{O}}{\tau} \tag{14.1.17}$$

$$\frac{d^{17}\mathrm{F}}{dt} = +\frac{^{16}\mathrm{O}}{\tau} - \frac{^{17}\mathrm{F}}{\tau_\beta} \tag{14.1.18}$$

$$\frac{d^{17}\mathrm{O}}{dt} = +\frac{^{17}\mathrm{F}}{\tau_\beta} - \frac{^{17}\mathrm{O}}{\tau} \tag{14.1.19}$$

where τ is the lifetime of the given species against reactions with a proton ($\tau = 1/\lambda f N_{^1\mathrm{H}}$), and τ_β is the species lifetime against beta (or positron) decay. In the CNO cycle, all the reactions, with the exception of $^{17}\mathrm{O}(p, {}^4\mathrm{He}){}^{14}\mathrm{N}$, are believed to be non-resonant.

There are several items of note in the previous equations. First, when ^{15}N combines with a proton, the result is an excited compound nucleus, ^{16}O*. This nucleus then decays into either normal ^{16}O or into ^{12}C + ^{4}He. The probability of the former reaction depends on the initial energy of ^{16}O, which, of course, depends on the energy of the reaction. Thus, the branching ratio, γ, which appears in (14.1.11) and (14.1.17), is actually a mean value that is averaged over the Maxwellian velocity distribution. It is thus dependent (weakly) on temperature. Its value, $\gamma \approx 4 \times 10^{-4}$, indicates that most CNO processing occurs through the first upper part of the bi-cycle.

A second item to note is that ^{13}N, ^{15}O, and ^{17}F can be lost either by beta decay or by reactions with a proton. In all cases, however, the timescales for beta decay are much shorter than that for nuclear fusion (typically seconds vs. years). Thus, unless the temperatures are extremely high ($T > 10^8$ K), the fusion terms in these equations can be dropped.

[Note: in the late stages of stellar evolution, temperatures can reach 10^8 K; when this happens, these additional reactions must be included. They are ^{13}N$(p, \gamma)^{14}$O, ^{15}O$(p, \gamma)^{16}$F, and ^{17}F$(p, \gamma)^{18}$Ne. In each case, the product quickly decays via a positron emission: ^{14}O $\rightarrow$ ^{14}N, ^{16}F $\rightarrow$ ^{16}O, and ^{18}Ne $\rightarrow$ ^{18}F $\rightarrow$ ^{18}O. If any ^{18}O is created, it will then combine with a proton and produce a ^{4}He nucleus via the reaction ^{18}O$(p, \alpha)^{15}$N.]

A corollary to the above is that because the beta decay rates are so short, ^{13}N, ^{15}O, and ^{17}F quickly reach equilibrium (in a matter of minutes). Thus, the time derivatives of (14.1.12), (14.1.15), and (14.1.18) can be set to zero, and these variables can be eliminated from the network. The calculation of the CNO cycle then reduces to solving six coupled differential equations:

$$\frac{d\,^{12}\mathrm{C}}{dt} = -\frac{^{12}\mathrm{C}}{\tau} + (1-\gamma)\,\frac{^{15}\mathrm{N}}{\tau} \qquad (14.1.20)$$

$$\frac{d\,^{13}\mathrm{C}}{dt} = +\frac{^{12}\mathrm{C}}{\tau} - \frac{^{13}\mathrm{C}}{\tau} \qquad (14.1.21)$$

$$\frac{d\,^{14}\mathrm{N}}{dt} = +\frac{^{13}\mathrm{C}}{\tau} - \frac{^{14}\mathrm{N}}{\tau} + \frac{^{17}\mathrm{O}}{\tau} \qquad (14.1.22)$$

$$\frac{d\,^{15}\mathrm{N}}{dt} = +\frac{^{14}\mathrm{N}}{\tau} - \frac{^{15}\mathrm{N}}{\tau} \qquad (14.1.23)$$

$$\frac{d\,^{16}\mathrm{O}}{dt} = \gamma\,\frac{^{15}\mathrm{N}}{\tau} - \frac{^{16}\mathrm{O}}{\tau} \qquad (14.1.24)$$

$$\frac{d\,^{17}\mathrm{O}}{dt} = +\frac{^{16}\mathrm{O}}{\tau} - \frac{^{17}\mathrm{O}}{\tau} \qquad (14.1.25)$$

Again, these equations can be solved numerically via Runge-Kutta integration. However, under most circumstances, the problem can be simplified enormously. Examine the table (again, from Clayton 1968), which gives the reaction lifetimes as a function of burning temperature, in terms of

$$\log\left(\frac{\rho X}{100}\tau_p\right)$$

with ρ in g-cm^{-3} and τ_p in years. For the higher temperatures (which correspond to higher mass stars), the timescales listed are all much shorter than the timescale for the depletion of hydrogen, or for changes in the stellar temperature and density. Thus, while the CNO isotopes are achieving equilibrium, it is fair to assume that the hydrogen abundance and main sequence structure of the star are

constant. This is important, since it allows us to fix the values of τ in equations (14.1.26) - (14.1.31), and linearize the equations.

To see this, let the subscripts 1 through 6 represent ^{12}C, ^{13}C, ^{14}N, ^{15}N, ^{16}O, and ^{17}O, and let y be the number the density of the species. In differential form for back-differencing, equations (14.1.20) through (14.1.25) are

$$\frac{y_1' - y_1}{\Delta t} = -\frac{y_1'}{\tau_1} + (1 - \gamma)\frac{y_4'}{\tau_4} \tag{14.1.26}$$

$$\frac{y_2' - y_2}{\Delta t} = +\frac{y_1'}{\tau_1} - \frac{y_2'}{\tau_2} \tag{14.1.27}$$

$$\frac{y_3' - y_3}{\Delta t} = +\frac{y_2'}{\tau_2} - \frac{y_3'}{\tau_3} + \frac{y_6'}{\tau_6} \tag{14.1.28}$$

$$\frac{y_4' - y_4}{\Delta t} = +\frac{y_3'}{\tau_3} - \frac{y_4'}{\tau_4} \tag{14.1.29}$$

$$\frac{y_5' - y_5}{\Delta t} = +\gamma\frac{y_4'}{\tau_4} - \frac{y_5'}{\tau_5} \tag{14.1.30}$$

$$\frac{y_6' - y_6}{\Delta t} = +\frac{y_5'}{\tau_5} - \frac{y_6'}{\tau_6} \tag{14.1.31}$$

These equations can now be re-written into a more useful form:

$$\left(1 + \frac{\Delta t}{\tau_1}\right) y_1' - \frac{(1 - \gamma)\Delta t}{\tau_4} y_4' = y_1 \qquad (14.1.32)$$

$$-\frac{\Delta t}{\tau_1} y_1' + \left(1 + \frac{\Delta t}{\tau_2}\right) y_2' = y_2 \qquad (14.1.33)$$

$$-\frac{\Delta t}{\tau_2} y_2' + \left(1 + \frac{\Delta t}{\tau_3}\right) y_3' - \frac{\Delta t}{\tau_6} y_6' = y_3 \qquad (14.1.34)$$

$$-\frac{\Delta t}{\tau_3} y_3' + \left(1 + \frac{\Delta t}{\tau_4}\right) y_4' = y_4 \qquad (14.1.35)$$

$$-\frac{\gamma \Delta t}{\tau_4} y_4' + \left(1 + \frac{\Delta t}{\tau_5}\right) y_5' = y_5 \qquad (14.1.36)$$

$$-\frac{\Delta t}{\tau_5} y_5' + \left(1 + \frac{\Delta t}{\tau_6}\right) y_6' = y_6 \qquad (14.1.37)$$

which, in matrix form is

$$\begin{pmatrix} X & 0 & 0 & X & 0 & 0 \\ X & X & 0 & 0 & 0 & 0 \\ 0 & X & X & 0 & 0 & X \\ 0 & 0 & X & X & 0 & 0 \\ 0 & 0 & 0 & X & X & 0 \\ 0 & 0 & 0 & 0 & X & X \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \\ y_6' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix}$$

where the X's represent the terms of (14.1.32) - (14.1.37). The updated CNO abundances can therefore be found via a simple 6×6 matrix inversion.

The time it takes the CNO cycle to reach equilibrium is a critical function of temperature. However, two properties of the chain should be noted. First, the CNO equations really consist of two separate cycles: a CN cycle and an ON cycle. If you examine the lifetimes of the CN reactions, you can see that this cycle proceeds fairly rapidly, with the timescale for ^{14}N fusion ($\tau \sim 10^6$ years at $T_6 \sim 20$ K) being the limiting factor. Since the branching ratio out of this cycle is very small ($\gamma \sim 4 \times 10^{-4}$), $\sim 10^3$ cycles are required before a significant number of CN nuclei are lost to the ON chain. Meanwhile, the timescale for ON nuclei to filter into the CN cycle is similarly long, $\tau \sim 10^8$ years (the ^{16}O reaction timescale for $T_6 \sim 20$ K). As a result, the CN cycle falls into quasi-equilibrium fairly quickly, and is largely decoupled from the slower ON cycle.

The second thing to note is that in high mass stars (*i.e.*, stars with high central temperatures), the reaction timescales are short, and the cycle achieves equilibrium fairly quickly. Thus, for purposes of estimating energy generation, the equilibrium abundances of the elements can (most times) be used. On the other hand, in low mass stars, where CNO equilibrium may never be achieved, the proton-proton chain dominates energy production. Thus, small errors in the CNO reaction are acceptable.

Of course, the operation of the CNO cycle means that the differential equations for ^{1}H and ^{4}He must be modified to include the additional destruction and creation mechanisms.

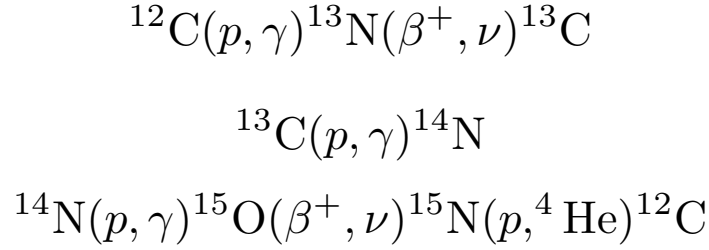**Table 5-3   Dependence of** $\log (\tau \rho X_H / 100)$ **on temperature**†

Temperature, T_6	$C^{12}(p,\gamma)N^{13}$	$C^{13}(p,\gamma)N^{14}$	$N^{14}(p,\gamma)O^{15}$	$N^{15}(p,\alpha)C^{12}$	$10^4\gamma$	$O^{16}(p,\gamma)F^{17}$	$O^{17}(p,\alpha)N^{14}$
5	16.32	15.73	19.79	15.53	4.649	22.95	21.92
6	14.32	13.73	17.57	13.29	4.598	20.51	20.02
7	12.72	12.13	15.79	11.50	4.551	18.56	18.26
8	11.41	10.81	14.32	10.03	4.508	16.95	16.50
9	10.29	9.69	13.08	8.78	4.468	15.59	15.10
10	9.33	8.73	12.02	7.70	4.431	14.42	14.05
11	8.50	7.90	11.09	6.76	4.396	13.39	13.15
12	7.75	7.15	10.26	5.93	4.363	12.49	12.38
13	7.09	6.49	9.52	5.18	4.332	11.68	11.68
14	6.49	5.89	8.86	4.51	4.303	10.95	11.02
15	5.95	5.35	8.26	3.90	4.275	10.29	10.32
16	5.45	4.85	7.71	3.34	4.248	9.68	9.55
17	5.00	4.39	7.20	2.83	4.223	9.13	8.70
18	4.58	3.97	6.73	2.35	4.198	8.61	7.86
19	4.18	3.58	6.30	1.91	4.175	8.14	7.01
20	3.82	3.21	5.89	1.50	4.152	7.69	6.18
22	3.16	2.55	5.16	0.75	4.110	6.89	4.78
24	2.57	1.97	4.51	0.09	4.071	6.18	3.63
25	2.30	1.70	4.21	−0.21	4.052	5.85	3.10
26	2.05	1.44	3.93	−0.50	4.034	5.54	2.62
28	1.58	0.97	3.41	−1.03	4.000	4.97	1.75
30	1.15	0.54	2.93	−1.51	3.967	4.45	1.05
35	0.23	−0.38	1.91	−2.55	3.893	3.33	−0.42
40	−0.53	−1.14	1.07	−3.42	3.829	2.41	−1.50
45	−1.18	−1.78	0.36	−4.14	3.771	1.64	−2.33
50	−1.73	−2.33	−0.25	−4.77	3.719	0.97	−2.99
55	−2.21	−2.82	−0.78	−5.32	3.673	0.39	−3.53
60	−2.64	−3.24	−1.25	−5.81	3.630	−0.12	−3.97
65	−3.02	−3.63	−1.67	−6.24	3.590	−0.58	−4.33
70	−3.37	−3.97	−2.05	−6.63	3.554	−0.99	−4.65
75	−3.68	−4.28	−2.39	−6.99	3.521	−1.37	−4.91
80	−3.97	−4.57	−2.71	−7.32	3.489	−1.71	−5.14
85	−4.23	−4.83	−2.99	−7.62	3.460	−2.02	−5.35
90	−4.48	−5.08	−3.26	−7.90	3.433	−2.31	−5.52
95	−4.70	−5.30	−3.51	−8.15	3.407	−2.58	−5.68
100	−4.91	−5.51	−3.74	−8.39	3.383	−2.83	−5.82

† Adapted from G. R. Caughlan and W. A. Fowler, *Astrophys. J.*, **136**:453 (1962). By permission of The University of Chicago Press. Copyright 1962 by The University of Chicago.

‡ The lifetimes against protons are expressed in years, and the density ρ is in grams per cubic centimeter.

Energy Generation in the CNO Cycle

The energy produced by the CNO cycle is simply the sum from the individual reactions in the chain. However, we can approximate the energy generation in equilibrium by neglecting the ON chain, and considering that the "cycle time" for CN is dominated by the slowest reactions. Thus, we can approximate the chain with three steps:

$$^{12}\mathrm{C}(p,\gamma)^{13}\mathrm{N}(\beta^+,\nu)^{13}\mathrm{C}$$

$$^{13}\mathrm{C}(p,\gamma)^{14}\mathrm{N}$$

$$^{14}\mathrm{N}(p,\gamma)^{15}\mathrm{O}(\beta^+,\nu)^{15}\mathrm{N}(p,^4\mathrm{He})^{12}\mathrm{C}$$

The first reaction produces $1.944 + 2.221 - 0.710 = 3.455$ MeV per reaction; the second generates 7.550 MeV per reaction, and the third 14.019 MeV, per reaction. The total energy for the chain in equilibrium is therefore

$$\rho\epsilon_{CN} \approx 5.534\,\frac{N(^{12}\mathrm{C})}{\tau_{12}} + 12.093\,\frac{N(^{13}\mathrm{C})}{\tau_{13}} + 22.453\,\frac{N(^{14}\mathrm{N})}{\tau_{14}}$$

$$\times\,10^{-6}\ \mathrm{ergs\ s^{-1}\ cm^{-3}} \qquad (14.2.1)$$

or, since the abundance of the individual isotope is simply proportional to its lifetime.

$$\rho\epsilon_{CN} \approx 40.08 \times 10^{-6}\,\frac{N(\mathrm{CN})}{\tau_{12}+\tau_{13}+\tau_{14}}\ \mathrm{ergs\ s^{-1}\ cm^{-3}} \qquad (14.2.2)$$

Now recall that these are non-resonant reactions. The nuclear lifetimes will depend on the reaction rates, λ, which has a temperature dependence of

$$\nu = \frac{d\ln\langle\sigma v\rangle}{d\ln T} = \frac{14.16}{T_6^{1/3}}\left\{Z_a^2 Z_X^2 A\right\}^{1/3} - \frac{2}{3} \qquad (11.14)$$

Note that this dependence is proportional to $Z^{2/3}$. Thus, we can expect that the CNO cycle's energy generation will be ~ 5 times more sensitive to temperature than the proton-proton.
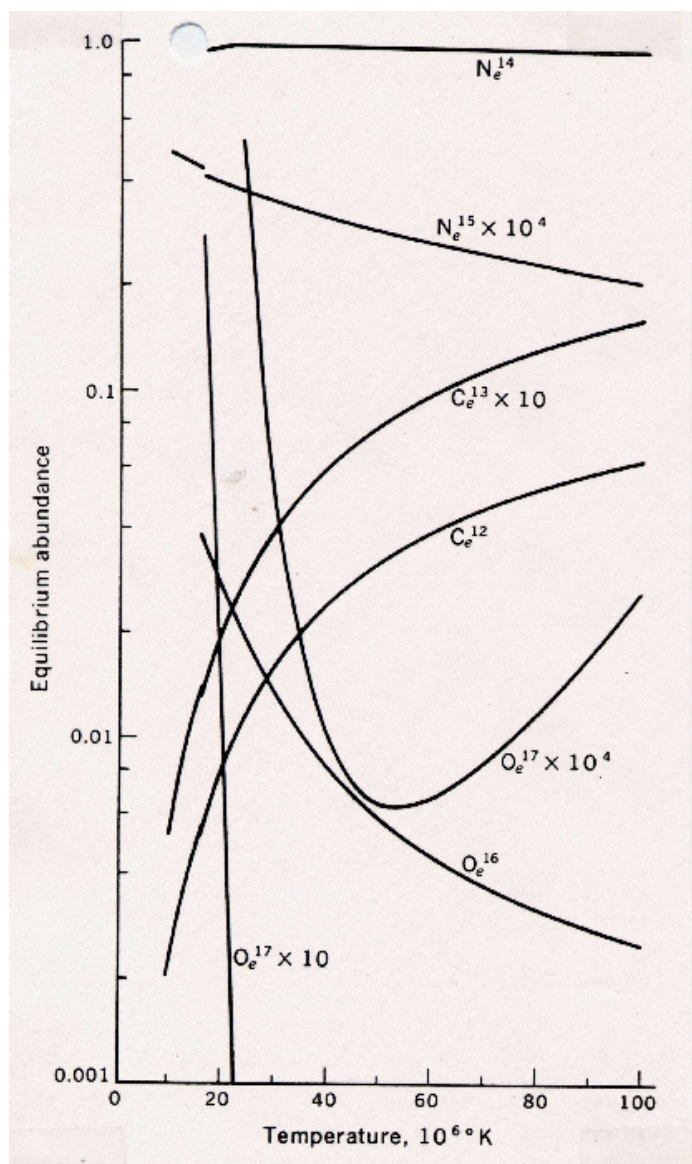
A reasonable approximation to the CNO cycle's rate of energy production in equilibrium is

$$\epsilon_{\text{CNO}} = 8.67 \times 10^{27}\, g_{14,1}\, X_{\text{CNO}}\, X\, \rho\, T_6^{-2/3}\, e^{-152.28/T_6^{1/3}} \text{ ergs s}^{-1}\text{ cm}^{-3}$$
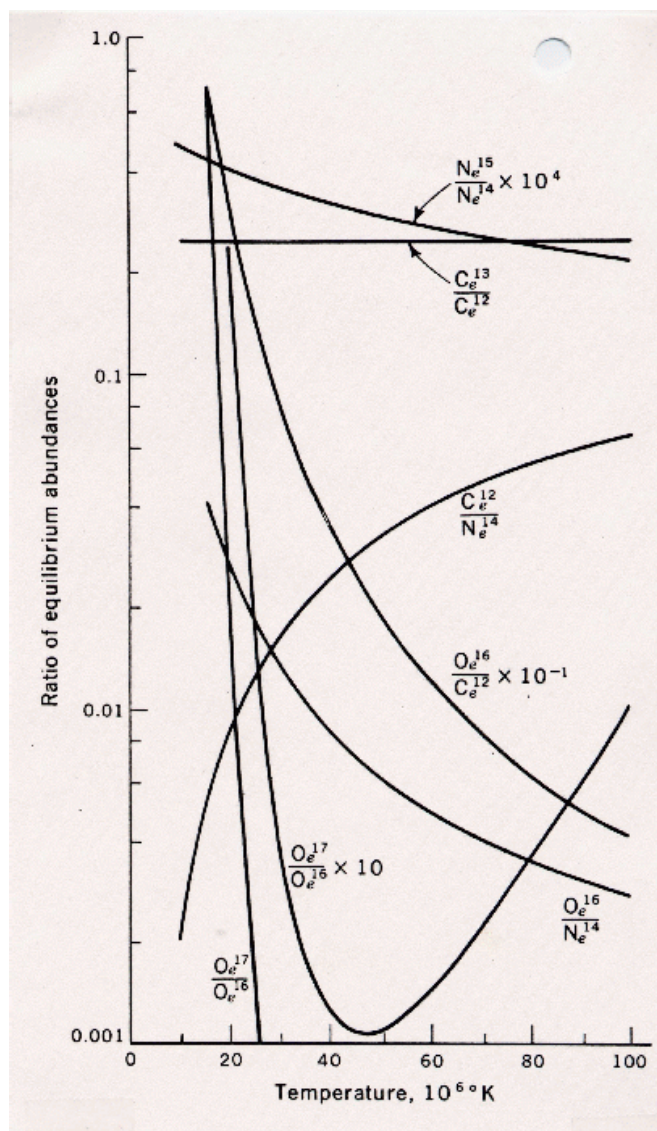
with

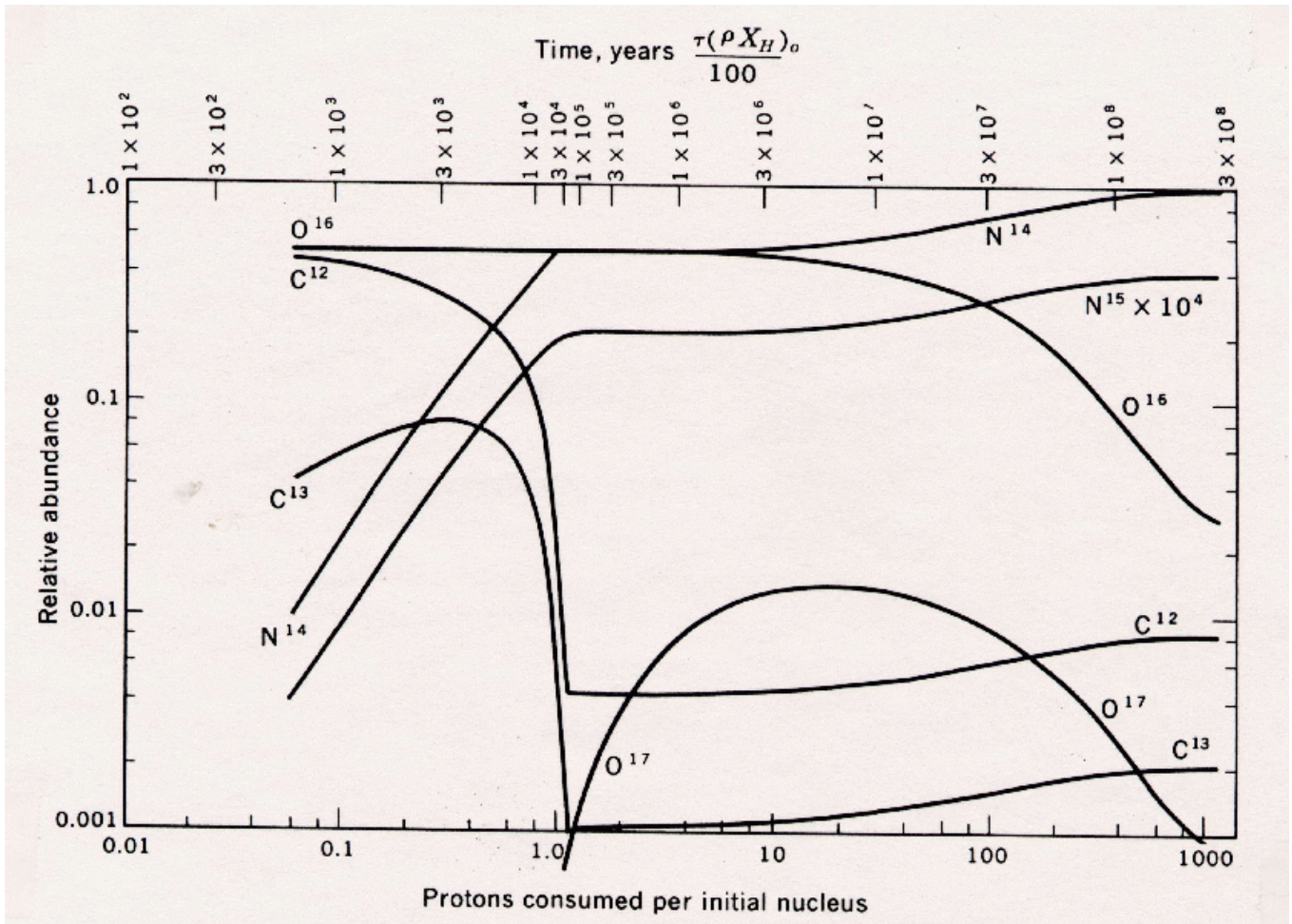$$g_{14,1} = 1 + 0.0027\, T_6^{1/3} - 0.00778\, T_6^{2/3} - 0.000149\, T_6$$

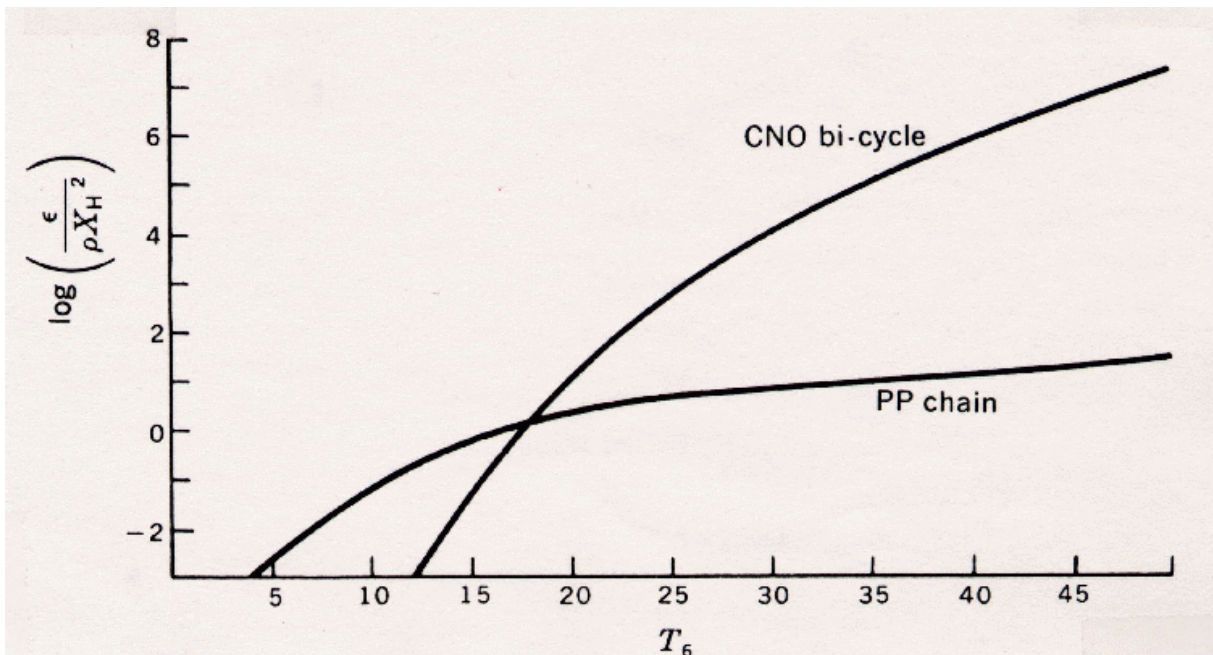As T_6 varies from 10 to 50, ν goes from 23 to 13. This compares to $\nu \sim 4$ for the proton-proton chain.

CNO Equilibrium Abundances

CNO Equilibrium Isotope Ratios

Time, years $\dfrac{\tau(\rho X_H)_0}{100}$

1×10^2 3×10^2 1×10^3 3×10^3 1×10^4 3×10^4 1×10^5 3×10^5 1×10^6 3×10^6 1×10^7 3×10^7 1×10^8 3×10^8

Relative abundance

1.0

O^{16}
C^{12}
N^{14}
$N^{15} \times 10^4$

0.1

C^{13}
O^{16}

0.01

N^{14}
C^{12}
O^{17}
C^{13}

0.001

O^{17}

0.01   0.1   1.0   10   100   1000

Protons consumed per initial nucleus

CNO approach to equilibrium starting with 50% ^{12}C, 50% ^{16}O.

$\log\left(\dfrac{\epsilon}{\rho X_H^{\,2}}\right)$

8
6
4
2
0
−2

CNO bi-cycle

PP chain

5   10   15   20   25   30   35   40   45

T_6

CNO vs. PP energy generation for $X_{CN}/X_H = 0.02$.

Numerical Note: Forward and Back Differencing Techniques

There are two ways to numerically solve a system of differential equations. The first method is through "forward differencing," or "explicit" numerical integration. With this technique, values for integrating step $n + 1$ are estimated directly from the known values of the previous step. In the alternative "back differencing" or "implicit" integration method, the values at step $n + 1$ are estimated using the $n + 1$ values themselves, via numerical inversion.

For example, consider the simple differential equation

$$\frac{dX}{dt} = a - bX$$

which has the analytical solution

$$X = \frac{a}{b} + e^{-bt}$$

In the forward differencing method, this equation is

$$\frac{X_{n+1} - X_n}{\Delta t} = a - bX_n$$

which translates to

$$X_{n+1} = a\Delta t + X_n \left(1 - b\Delta t\right) \tag{4.2.1}$$

while in the back differencing method, the equation is

$$\frac{X_{n+1} - X_n}{\Delta t} = a - bX_{n+1}$$

or

$$X_{n+1} = \frac{a\Delta t + X_n}{1 + b\Delta t} \tag{4.2.2}$$

Now note that (4.2.1) and (4.2.2) behave very differently as the time step becomes large, *i.e.*, $\Delta t \to \infty$. With forward differencing, the numerical solution diverges; with back differencing, $X_{n+1} \to a/b$, which is the analytical result. This is characteristic of the two approaches. In general, forward differencing solutions are unstable, and can only be used with very small time steps (*i.e.*, for the above example, $b\Delta t < 1$). Back differencing methods are much more reliable, and should be used whenever possible.